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STRESSES AROUND A CIRCULAR HOLE IN A SHALLOW CONICAL SHELL WITH TORSIONAL LOADING

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Abstract—Analytical solutions are presented for the stresses in a conical shell having a circular hole on its lateral surface. The shell is subjected to torsional load. The method of analysis involves perturbations in parameters defining curvature and the cone angle of the shell (β and ε respectively). The membrane and bending stresses are obtained retaining terms of the order of β^4 and ε^2.

1. INTRODUCTION

THE stresses around a circular hole in a conical shell have been determined with the axial tension and internal pressure loading retaining terms of β^2 and ε^2 order [1]. It has been noticed [1] that retaining terms of ε^2 order is not very useful in the sense that conical shell solutions are not very much different from cylindrical ones, unless higher order terms in β are also retained. It appears from the correlation between these parameters ε and β that ε is of β^2 order provided $r_0/h \geq 2{\cdot}42 \tan\alpha(\mu = 0{\cdot}3)$. In view of the above correlation between the orders of these parameters, it becomes essential to consider the terms of β^4 order at least, if one decides to consider the ε^2 order terms. This has motivated the present investigation, where solutions are attempted, retaining terms of β^4 and ε^2 order for the torsional loading. Formulae, from which the membrane and bending stresses can be computed, are presented and numerical results are given for various values of these parameters.

2. THE GOVERNING EQUATIONS

2.1. *The differential equation*

Notation is the same as used in [1].§ The differential equation for a thin conical shell is obtained from [1] as follows:

$$\nabla^4\phi + \frac{8i\beta^2}{\varepsilon s}\frac{\partial^2\phi}{\partial s^2} = 0. \tag{1}$$

2.2. *Boundary conditions*

The boundary conditions at $r = 1$ are

$$N_r^T = 0, \qquad N_{r\theta}^T = 0$$
$$M_r = 0, \qquad Q_r^* = 0. \tag{2}$$

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§ In [1] Poisson's ratio has been denoted by ν, but in the present Paper it is denoted by μ.

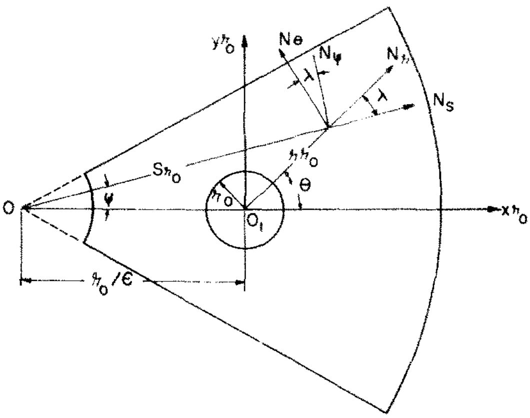

FIG. 1(a). Conical shell developed on a flat surface.

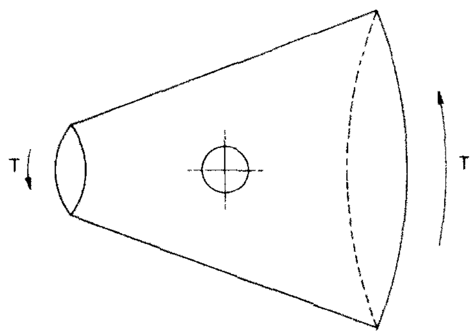

FIG. 1(b). Conical shell subjected to torsional load T.

3. METHOD OF ANALYSIS

The method of analysis involves perturbations with respect to the parameter β and ε. The hole is assumed to be small enough so that $\beta < 1$. It can be shown that ε is of the order of β^2 if

$$\frac{r_0}{h} \geq 2{\cdot}42 \tan \alpha \, (\mu = 0{\cdot}3).$$

Now the complex stress–displacement function and all the stress resultants can be expressed in power series of ε.

$$\begin{aligned} \phi &= \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 \\ N_r &= N_{r0} + \varepsilon N_{r1} + \varepsilon^2 N_{r2} \qquad \text{etc.} \end{aligned} \tag{3}$$

Further, each term in the power series can be expressed in even powers of β, and products of even powers of β and ln β. For instance,

$$\phi_j = \phi_{j0} + \beta^2 \ln \beta\phi_{j1} + \beta^2\phi_{j2} + \beta^4 \ln \beta\phi_{j3} + \beta^4\phi_{j4}. \tag{4}$$

In this paper, we are retaining the terms of the order of β^4. The terms of $\beta^6 \ln \beta$ order and higher have been neglected. Substituting $j = 0$, 1 and 2, in the above expression,

$$\begin{aligned} \phi_0 &= \phi_{00} + \beta^2 \ln \beta \phi_{01} + \beta^2 \phi_{02} + \beta^4 \ln \beta \phi_{03} + \beta^4 \phi_{04} \\ \phi_1 &= \phi_{10} + \beta^2 \ln \beta \phi_{11} + \beta^2 \phi_{12} + 0(\beta^4 \ln \beta) \\ \phi_2 &= \phi_{20} + 0(\beta^2 \ln \beta). \end{aligned} \tag{5}$$

As ε is of the order of β^2, we can neglect the term of the order of $\beta^4 \ln \beta$ in the expansion of ϕ_1 and terms of the order of $\beta^2 \ln \beta$ in the expansion of ϕ_2.

4. SOLUTION FOR SMALL VALUES OF β AND ε

The membrane and bending solution will be obtained by considering only the first three terms of series in the expression (3). For this approximation, equation (1) reduces to three differential equations for ϕ_0, ϕ_1, ϕ_2. These equations are the same as equations (11a), (11b) and (11c) of [1]. The complementary solution of any of these equations is given below:

$$\phi_j = \sum_{n=1}^{\infty} (A_{n1} + iB_{n1})(\alpha_{n1} + i\beta_{n1}) + \sum_{n=1}^{\infty} (A_{n2} + iB_{n2})(\alpha_{n2} + \beta_{n2}) \tag{6}$$

where

$$\begin{aligned} \alpha_{n1} + i\beta_{n1} &= \cosh[(1-i)\beta x] H_n^1 (\sqrt{2i}\, \beta r) \sin n\theta \\ \alpha_{n2} + i\beta_{n2} &= \sinh[(1-i)\beta x] H_n^1 (\sqrt{2i}\, \beta r) \sin n\theta. \end{aligned} \tag{7}$$

4.1. *Modified boundary conditions*

The boundary conditions as defined in Section 2.2 are reformulated in this section. The membrane boundary conditions at $r = 1$ are formulated as follows:—

$$\begin{aligned} N_{rj} + \bar{N}_{rj} &= 0 \\ N_{r\theta j} + \bar{N}_{r\theta j} &= 0 \end{aligned} \tag{8}$$

where $j = 0, 1, 2$.

Substituting for N_{rj} and $N_{r\theta j}$ from [1] and $\bar{N}_{rj}$ and $\bar{N}_{r\theta j}$ from Appendix in the above equations, following boundary conditions for zeroth, first and second order approximation in ε are obtained

Zeroth approximation:

$$\begin{aligned} \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \operatorname{Im} \phi_{0k} & \begin{cases} = m r_0^2 \tau_0 \sin 2\theta, & k = 0 \\ = 0, & k = 1, 2, 3, 4 \ldots \end{cases} \\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) \operatorname{Im} \phi_{0k} & \begin{cases} = -m r_0^2 \tau_0 \cos 2\theta, & k = 0 \\ = 0, & k = 1, 2, 3, 4 \ldots . \end{cases} \end{aligned} \tag{9}$$

First approximation:

$$\begin{aligned} \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \operatorname{Im} \phi_{1k} & \begin{cases} = -2 m r_0^2 \tau_0 \sin 3\theta, & k = 0 \\ = 0, & k = 1, 2, \end{cases} \\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) \operatorname{Im} \phi_{1k} & \begin{cases} = 2 m r_0^2 \tau_0 \cos 3\theta, & k = 0 \\ = 0, & k = 1, 2. \end{cases} \end{aligned} \tag{10}$$

Second approximation:

$$\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\operatorname{Im}\phi_{20} = 3\tau_0 m r_0^2 \sin 4\theta$$
$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\right]\operatorname{Im}\phi_{20} = -3\tau_0 m r_0^2 \cos 4\theta. \tag{11}$$

Using the formula of M_r and Q_r^* from [1], the boundary conditions for bending at $r = 1$ are recorded below:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{\mu}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\mu}{r}\frac{\partial}{\partial r}\right)\operatorname{Re}\phi_{jk} = 0$$
$$\left[\frac{\partial}{\partial r}\nabla^2+\frac{1-\mu}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2}{\partial\theta^2}\right)\right]\operatorname{Re}\phi_{jk} = 0 \tag{12}$$

$j = 0, 1, 2$

$k = 0, 1, 2, 3, 4.$

5. ZEROTH APPROXIMATION

This approximation corresponds to cylindrical shell solution. The complex stress function ϕ_0 is determined by the use of appropriate boundary conditions. The expression for ϕ_0 is

$$\phi_0 = \phi_{00}+\phi_{01}\beta^2\ln\beta+\phi_{02}\beta^2+\phi_{03}\beta^4\ln\beta+\phi_{04}\beta^4$$

where

$$\phi_{00} = 2i\frac{\sin 2\theta}{\pi}A_{12}^0\left(1-\frac{1}{2r^2}\right)$$

$$\phi_{01} = \frac{2\sin 2\theta}{\pi}A_{12}^0\left(\frac{1-\mu}{3+\mu}\right)\left[-2+\frac{1}{r^2}-\frac{3+\mu}{1-\mu}r^2\right]$$

$$\operatorname{Im}\phi_{02} = \frac{\sin 2\theta}{18}A_{12}^0\left[\frac{1}{r^2}+9r^2-6\right]$$

$$\operatorname{Re}\phi_{02} = \frac{2\sin 2\theta}{\pi}\left[-\frac{A_{21}^3}{r^2}+B_{12}^1-\frac{4A_{32}^3}{r^2}+\frac{A_{12}^0}{3}\left(1-3\ln\frac{\gamma r}{\sqrt{2}}\right)\right]$$
$$-\frac{\sin 4\theta}{\pi}\left[\frac{24B_{41}^5}{r^4}+\frac{8A_{32}^3}{r^2}+\frac{B_{21}^1}{2}+\frac{A_{12}^0 r^2}{6}\right]$$

$$\operatorname{Im}\phi_{03} = -\frac{2\sin 2\theta}{\pi}\left[\frac{B_{21}^6}{r^2}+\frac{B_{21}^1}{r^2}-\frac{5}{12}A_{12}^0 r^4+A_{12}^4-\frac{B_{12}^2 r^2}{3}+B_{12}^2 r^2\ln\frac{\gamma r}{\sqrt{2}}+B_{12}^1 r^2+\frac{4B_{32}^6}{r^2}\right]$$
$$+\frac{\sin 4\theta}{\pi}\left[\frac{A_{21}^4}{2}+\frac{24A_{41}^8}{r^4}+\frac{A_{12}^0 r^4}{6}-\frac{B_{12}^2 r^2}{6}-\frac{8B_{32}^6}{r^2}\right]$$

$$\operatorname{Re}\phi_{03} = \frac{\sin 2\theta}{\pi}\left[-\frac{2A_{21}^6}{r^2}-2A_{12}^1 r^2+2B_{12}^4-\frac{\pi B_{12}^2}{2}r^2\right]$$

$$\operatorname{Im}\phi_{04} = \frac{\sin 2\theta}{\pi}\left[-\frac{2B_{21}^5}{r^2} - r^2 B_{21}^1\left(\frac{1}{48}+\frac{1}{2}\ln\frac{\gamma r}{\sqrt{2}}\right) + \frac{6B_{41}^5}{r^2} + \frac{\pi}{2}A_{12}^1 r^2 - 2A_{12}^3 + \frac{2}{3}B_{12}^1\right.$$

$$\left.\times\left(1-3\ln\frac{\gamma r}{\sqrt{2}}\right) - \frac{8B_{32}^5}{r^2} + \frac{A_{12}^0 r^4}{240}\left(-185+200\ln\frac{\gamma r}{\sqrt{2}}\right)\right] + \frac{\sin 4\theta}{\pi}\left[\frac{A_{21}^3}{2} - \frac{B_{21}^1 r^2}{6}\right.$$

$$\left. + \frac{24A_{41}^7}{r^4} + \frac{8B_{41}^5}{r^2} + \frac{A_{12}^6 r^4}{240}\left(16+40\ln\frac{\gamma r}{\sqrt{2}}\right) - \frac{B_{12}^1 r^2}{6} - \frac{8B_{32}^5}{r^2} + \frac{192A_{52}^9}{r^4}\right] + \frac{\sin 6\theta}{\pi}$$

$$\times\left[\frac{B_{21}^1 r^2}{48} + \frac{6B_{41}^5}{r^2} + \frac{A_{12}^0 r^4}{240} + \frac{2}{3}A_{32}^3 + \frac{192}{r^4}A_{52}^9 - \frac{960}{r^6}A_{61}^4\right]$$

$$\operatorname{Re}\phi_{04} = \frac{\sin 2\theta}{\pi}\left[-\frac{2A_{21}^5}{r^2} - \frac{\pi B_{21}^1}{8}r^2 + \frac{5\pi}{24}A_{12}^0 r^4 + \frac{2}{3}A_{12}^1 r^2\right.$$

$$\left. - 2A_{12}^1 r^2 \ln\frac{\gamma r}{\sqrt{2}} - \frac{\pi B_{12}^1}{2}r^2 + 2B_{12}^3 - \frac{8A_{32}^5}{r^2}\right]$$

$$+\sin 4\theta\left[-\frac{B_{21}^3}{2} - \frac{24B_{41}^7}{r^4} + \frac{\pi A_{12}^0 r^4}{24} - \frac{A_{12}^1 r^2}{6} - \frac{8A_{32}^5}{r^2}\right]$$

where, the different A^s and B^s with subscript and superscript are given below:

$$A_{12}^0 = \frac{m\pi}{2}r_0^2\tau_0, \qquad A_{12}^1 = \frac{\pi}{6}A_{12}^0, \qquad A_{21}^4 = \frac{\mu-1}{\mu+3}A_{12}^0$$

$$A_{32}^3 = \frac{A_{12}^0}{12(3+\mu)}, \qquad A_{21}^3 = A_{12}^0\left[\frac{3\mu^2-6\mu-1}{6(3+\mu)(\mu-1)} + \frac{4\mu}{3(3+\mu)}\ln\frac{\gamma}{\sqrt{2}}\right]$$

$$A_{41}^7 = \frac{A_{12}^0}{480}\left[\frac{10(9\mu-1)}{3(3+\mu)}\ln\frac{\gamma}{\sqrt{2}} + \frac{4034\mu^2+11{,}860\mu-18{,}330}{360(3+\mu)(\mu-1)}\right]$$

$$A_{12}^3 = \frac{A_{12}^0}{8}\left[-\frac{2\pi^2}{3} + \frac{18(5-\mu)}{3+\mu}\left(\ln\frac{\gamma}{\sqrt{2}}\right)^2 - \frac{58\mu+304}{3(3+\mu)}\ln\frac{\gamma}{\sqrt{2}} - \frac{948\mu^2-2141\mu+17{,}809}{360(3+\mu)(\mu-1)}\right]$$

$$A_{52}^9 = \frac{1-\mu}{3+\mu}\cdot\frac{A_{12}^0}{24\times 384}, \qquad A_{61}^4 = -\frac{A_{12}^0}{42\times 960}\,\frac{63\mu+49}{240(\mu+3)}$$

$$B_{21}^3 = -\frac{\pi}{36}A_{12}^0, \qquad B_{21}^1 = -\frac{A_{12}^0}{2}, \qquad B_{12}^2 = -\frac{2(1-\mu)}{3+\mu}A_{12}^0$$

$$B_{12}^1 = A_{12}^0\left[\frac{\mu-15}{6(\mu+3)} + \frac{5-\mu}{3+\mu}\ln\frac{\gamma}{\sqrt{2}}\right], \qquad B_{41}^5 = \frac{A_{12}^0}{288}\,\frac{\mu-3}{\mu+3}$$

$$B_{32}^5 = \frac{A_{12}^0}{48}\left[\frac{79\mu^2+188\mu-327}{15(3+\mu)(\mu-1)} + \frac{3(5\mu-1)}{3+\mu}\ln\frac{\gamma}{\sqrt{2}}\right]$$

$$B_{21}^5 = \frac{A_{12}^0}{12}\left[\left(\ln\frac{\gamma}{\sqrt{2}}\right)^2\frac{10(5-\mu)}{3+\mu} - \ln\frac{\gamma}{\sqrt{2}}\,\frac{59\mu+357}{6(3+\mu)} - \frac{\pi^2}{2} + \frac{97\mu+279}{16(3+\mu)}\right].$$

5.1. *Membrane stresses*

The membrane stress $\sigma^T_{\theta 0}$ is expressed as

$$\sigma^T_{\theta 0} = \sigma^T_{\theta 00} + \beta^2 \sigma^T_{\theta 02} + \beta^4 \ln \beta \sigma^T_{\theta 03} + \beta^4 \sigma^T_{\theta 04}$$

where $\sigma^T_{\theta 0j}$ are determined from the formula

$$\sigma^T_{\theta 0j} = \overline{\sigma_{\theta 0j}} - \frac{1}{hmr_0^2} \frac{\partial^2}{\partial r^2} \operatorname{Im} \phi_{0j}.$$

The general expression for $\sigma^T_{\theta 0}$ is not given here as it is lengthy one. However, the expression at $r = 1$ is recorded as below:

$$\left[\frac{\sigma^T_{\theta 0}}{\tau_0^*}\right]_{r=1} = -4 \sin 2\theta - \frac{2\pi}{3} \sin 2\theta \beta^2 + \beta^4 \ln \beta \Bigg[\sin 2\theta \left\{ 8 \ln \frac{\gamma}{\sqrt{2}} + \frac{3\mu - 295}{6(\mu+3)} \right\}$$
$$- \frac{2(\mu+7)}{\mu+3} \sin 4\theta \Bigg] + \beta^4 \Bigg[\sin 2\theta \left\{ \frac{1522\mu^2 + 4004\mu - 6006}{240(3+\mu)(\mu-1)} + \frac{\pi^2}{6} - \frac{5\mu+151}{3(3+\mu)} \ln \frac{\gamma}{\sqrt{2}} \right.$$
$$\left. + \frac{5(5-\mu)}{3+\mu} \left(\ln \frac{\gamma}{\sqrt{2}} \right)^2 \right\} - \sin 4\theta \left\{ \frac{2448\mu^2 + 432\mu - 1578}{360(3+\mu)(\mu-1)} + \frac{71\mu - 3}{3(3+\mu)} \ln \frac{\gamma}{\sqrt{2}} \right\}$$
$$- \sin 6\theta \frac{1}{6(3+\mu)} \Bigg] \tag{13}$$

where $\tau_0^* = \tau_0/h$.

5.2. *Bending stresses*

The bending stress $\sigma_{\theta 0}$ is expressed as

$$\sigma_{\theta 0} = \beta^2 \ln \beta \sigma_{\theta 01} + \beta^2 \sigma_{\theta 02} + \beta^4 \ln \beta \sigma_{\theta 03} + \beta^4 \sigma_{\theta 04}$$

where $\sigma_{\theta 0j}$ are obtained from the formula

$$\sigma_{\theta 0j} = \frac{6}{h^2} \frac{D}{r_0^2} \left(\frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} + \mu \frac{\partial^2}{\partial r^2} \right) \operatorname{Re} \phi_{0j}.$$

The expression for $\sigma_{\theta 0}$ at $r = 1$ is given below:

$$\left(\frac{\sigma_{\theta 0}}{\tau_0^*}\right)_{r=1} = \frac{1}{2} \left(\frac{3}{1-\mu^2} \right)^{\frac{1}{2}} \Bigg[\sin 2\theta \frac{16(1-\mu^2)}{3+\mu} \beta^2 \ln \beta + \beta^2 \left\{ -2 \sin 2\theta \left(\frac{2\pi}{9} + \frac{4}{3} \frac{2\mu^2 + 3\mu + 9}{3+\mu} \right. \right.$$
$$\left. \left. + \frac{8(\mu-1)^2}{3+\mu} \ln \frac{\gamma}{\sqrt{2}} \right) + \sin 4\theta \frac{4}{3} \frac{\mu^2 - 10\mu + 9}{3+\mu} \right\} + \beta^4 \ln \beta 2\pi \sin 2\theta \frac{\mu(22+10\mu)}{9(3+\mu)}$$
$$+ \beta^4 \left(- \frac{\pi \sin 2\theta}{72(3+\mu)^2} \left\{ 248\mu^3 + 128\mu^2 - 1405\mu + 1329 + 48(\mu-1)(\mu^2 + 28\mu + 59) \right. \right.$$
$$\left. \left. \times \ln \frac{\gamma}{\sqrt{2}} \right\} + \frac{\pi \sin 4\theta}{18(3+\mu)} (71\mu^2 - 107\mu + 62) \right) \Bigg]. \tag{14}$$

6. FIRST APPROXIMATION

The governing equation for the approximation is

$$\nabla^4\phi_1 + 8i\beta^2\frac{\partial^2\phi_1}{\partial x^2} = \beta^2 iL_1\phi_0.$$

Replacing ϕ_0 in the above equation by its expansion, we obtain

$$\nabla^4\phi_1 + 8i\beta^2\frac{\partial^2\phi_1}{\partial x^2} = iL_1[\beta^2\phi_{00} + 0(\beta^4 \ln \beta)].$$

Since we are neglecting the terms of order of $\beta^4 \ln \beta$ in the expansion of ϕ_1, the particular solution of the above equation can also be terminated at β^2 order terms.

The solution of the above equation, ϕ_1 can be expressed as

$$\phi_1 = \phi_1^P + \phi_1^C$$

where ϕ_1^P is particular solution, and

ϕ_1^C is complementary solution.

ϕ_1^P can be assumed as $\phi_1^P = \phi_{12}^P\beta^2 + 0(\beta^4 \ln \beta)$.

Substituting the assumed expansion of ϕ_1^P in the governing equation and equating the coefficients of β^2, we have

$$\nabla^4\phi_{12}^P = iL_1\phi_{00}.$$

The above equation is rewritten after substituting ϕ_{00} from 0th approximation and operating it by the differential operator L_1 as defined earlier.

$$\nabla^4\phi_{12}^P = -\frac{48B_{21}^1}{\pi}[-3r^{-3}\sin 5\theta + r^{-3}\sin 3\theta] + \frac{16A_{12}^0}{\pi r}(3\sin 5\theta - 2\sin 3\theta - \sin\theta).$$

Solving the above equation,

$$\phi_{12}^P = \frac{B_{21}^1}{4\pi}r(\sin 5\theta - 3\sin 3\theta) + \frac{A_{12}^0}{\pi}\left[\frac{r^3}{8}\sin 5\theta + \frac{2}{3}r^3\ln r\sin 3\theta - \frac{r^3\sin\theta}{4}(4\ln r - 3)\right] \tag{15}$$

Complementary solution. The complementary solution ϕ_1^C is assumed as

$$\phi_1^C = \phi_{10}^C + \beta^2 \ln \beta\phi_{11}^C + \beta^2\phi_{12}^C$$

combining the assumed complementary solution and particular solutions, ϕ_1 is determined by using the appropriate boundary conditions.

$$\phi_1 = \phi_{10} + \beta^2 \ln \beta\phi_{11} + \beta^2\phi_{12}$$

where

$$\phi_{10} = i\frac{2\sin 3\theta}{\pi}A_{11}^2\left(\frac{1}{r} - \frac{2}{3r^3}\right)$$

$$\phi_{11} = 0$$

$$\text{Im}\,\phi_{12} = \frac{A_{11}^2 r}{2}\sin\theta$$

$$\operatorname{Re}\phi_{12} = \frac{\sin 5\theta}{\pi}\left[-\frac{2B_{31}^1}{r}-\frac{192B_{51}^3}{r^5}-\frac{24B_{42}^3}{r^3}+\frac{A_{12}^0r^3}{8}-\frac{B_{22}^1r}{12}+\frac{B_{21}^1r}{4}\right]$$
$$+\frac{\sin 3\theta}{\pi}\left[-\frac{8A_{31}^3}{r^3}-\frac{2B_{31}^1}{r}-\frac{2A_{22}^3}{r}-\frac{24B_{42}^3}{r^3}+\frac{2}{3}A_{12}^0r^3\ln r-\frac{A_{11}^2r}{2}+\frac{3}{4}B_{22}^1r-\frac{3}{4}B_{21}^1r\right]$$
$$+\frac{\sin\theta}{\pi}\left[\frac{2B_{11}^4}{r}-\frac{2B_{31}^1}{r}-\frac{2A_{22}^3}{r}-\frac{A_{12}^0}{4}r^3(4\ln r-3)+\frac{A_{11}^2}{2}r\left(1-4\ln\frac{\gamma r}{\sqrt{2}}\right)+\frac{5}{6}B_{22}^1r\right] \quad (16)$$

where

$$A_{11}^2 = \frac{m\pi}{2}r_0^2\tau_0(=A_{12}^0), \qquad A_{22}^3 = -\frac{A_{11}^2}{24}\frac{13\mu+7}{\mu+3}$$

$$A_{31}^3 = \frac{A_{11}^2}{30\times 96}\frac{180\mu^2+244\mu+176}{(3+\mu)(1-\mu)}, \qquad B_{31}^1 = \frac{A_{11}^2}{6}$$

$$B_{51}^3 = A_{11}^2\frac{1125\mu^2-1518\mu-183}{60(3+\mu)(1-\mu)\times 5760}, \qquad B_{22}^1 = -A_{11}^2$$

$$B_{42}^3 = \frac{73+125\mu}{15\times 192(3+\mu)}A_{11}^2, \qquad B_{11}^4 = A_{11}^2\frac{13\mu^2+22\mu+13}{12(3+\mu)(1-\mu)}.$$

6.1. *Membrane stresses*

The membrane stress $\sigma_{\theta 1}^T$ is expressed as

$$\sigma_{\theta_1}^T = \sigma_{\theta_{10}}^T+\beta^2\ln\beta\sigma_{\theta_{11}}^T+\beta^2\sigma_{\theta_{12}}^T$$

where $\sigma_{\theta_{ij}}^T$ is obtained from

$$\sigma_{\theta_{ij}}^T = \overline{\sigma_{\theta_{ij}}}-\frac{1}{hmr_0^2}\frac{\partial^2}{\partial r^2}\operatorname{Im}\phi_{ij}$$

where $j = 0, 1, 2$

The general expression of $\sigma_{\theta_1}^T$ is

$$\sigma_{\theta_1}^T = 2\tau_0^*\sin 3\theta\left(r+\frac{4}{r^5}-\frac{1}{r^3}\right). \quad (17)$$

Substituting $r = 1$ in the above equation, we obtain

$$\left(\frac{\sigma_{\theta_1}^T}{\tau_0^*}\right)_{r=1} = 8\sin 3\theta.$$

6.2. *Bending stresses*

The bending stress σ_{θ_1} is expressed as

$$\sigma_{\theta_1} = \beta^2\sigma_{\theta_{12}}.$$

The free term and the $\beta^2\ln\beta$ term vanish individually because the real parts of the corresponding complex function ϕ vanish.

The σ_{θ_1} at $r = 1$ is recorded below:

$$\left(\frac{\sigma_{\theta_1}}{\tau_0^*}\right)_{r=1} = \beta^2\left(\frac{3}{1-\mu^2}\right)^{\frac{1}{2}}\left[\sin 5\theta\frac{115\mu^2-144\mu-259}{15(3+\mu)}+\sin 3\theta\frac{-\mu^2+60\mu+61}{3(3+\mu)}\right.$$

$$\left.-(4\mu+5)\sin\theta\right]. \tag{18}$$

7. SECOND APPROXIMATION

The governing equation of ϕ_2 is

$$\nabla^4\phi_2+8i\beta^2\frac{\partial^2\phi_2}{\partial x^2} = i\beta^2(L_1\phi+L_2\phi_2). \tag{19}$$

The particular solution of the above equation will be starting from the term containing β^2, since β^2 order terms are not to be considered for this approximation, there is no need of determining the particular solution. The complementary solution also will be confined to free term i.e. ϕ_{20}.

ϕ_2 is recorded as below:

$$\phi_2 = i\frac{mr_0^2}{4}\tau_0\sin 4\theta\left(\frac{3}{r^4}-\frac{4}{r^2}\right). \tag{20}$$

The membrane stress $\sigma_{\theta_2}^T$ is now computed from

$$\sigma_{\theta_2}^T = \overline{\sigma_{\theta_2}}-\frac{1}{hmr_0^2}\frac{\partial^2}{\partial r^2}\operatorname{Im}\phi_2$$

and recorded as

$$\sigma_{\theta_2}^T = -\tau_0^*\sin 4\theta\left(\frac{15}{r^6}-\frac{6}{r^4}+3r^2\right).$$

Substituting $r = 1$ in the above expression,

$$\frac{\sigma_{\theta_2}^T}{\tau_0^*} = -12\sin 4\theta.$$

Bending stress is vanishing as $\operatorname{Re}\phi_2 = 0$.

8. COMPLETE SOLUTION

The complete solution to the problem up to the terms containing ε^2 is now written by adding 0th, 1st and 2nd approximation solution. The total stress σ_θ^T is expressed as the sum of total membrane and bending stresses namely

$$\sigma_\theta^T = \sigma_{\theta(m)}^T+\sigma_{\theta b}^T$$

where m and b denote membrane and bending solutions. Now $\sigma_{\theta_m}^T$ is obtained from the following

$$\sigma_{\theta_m}^T = \sigma_{\theta_{m0}}^T+\varepsilon\sigma_{\theta_{m1}}^T+\varepsilon^2\sigma_{\theta_{m2}}^T.$$

Substituting the $\sigma^T_{\theta_{m0}}, \sigma^T_{\theta_{m1}}$ and $\sigma^T_{\theta_{m2}}$ at $r = 1$ in the above expression, we get

$$\left(\frac{\sigma^T_{\theta_m}}{\tau_0^*}\right)_{r=1} = -4\sin 2\theta - \frac{2\pi}{3}\sin 2\theta\beta^2 + \beta^4 \ln\beta\left[\sin 2\theta\left\{8\ln\frac{\gamma}{\sqrt{2}} + \frac{3\mu-295}{6(3+\mu)}\right\} - \frac{2(\mu+7)}{3+\mu}\sin 4\theta\right] + \beta^4\left[\sin 2\theta\left\{\frac{1522\mu^2+4004\mu-6006}{240(3+\mu)(\mu-1)} + \frac{\pi^2}{6} - \frac{5\mu+151}{3(3+\mu)}\ln\frac{\gamma}{\sqrt{2}} + \frac{5(5-\mu)}{3+\mu}\left(\ln\frac{\gamma}{\sqrt{2}}\right)^2\right\} - \sin 4\theta\left\{\frac{2448\mu^2+432\mu-1578}{360(3+\mu)(\mu-1)} + \frac{71\mu-3}{3(3+\mu)}\ln\frac{\gamma}{\sqrt{2}}\right\} - \sin 6\theta\frac{1}{6(3+\mu)}\right] + 8\varepsilon\sin 3\theta - 12\varepsilon^2\sin 4\theta. \tag{21}$$

Similarly $\sigma^T_{\theta_b}$ is obtained from the following:

$$\sigma^T_{\theta_b} = \sigma^T_{\theta_{b0}} + \varepsilon\sigma^T_{\theta_{b1}} + \varepsilon^2\sigma^T_{\theta_{b2}}$$

As

$$\left(\frac{\sigma^T_{\theta_b}}{\tau_0^*}\right)_{r=1} = \left(\frac{3}{1-\mu^2}\right)^{\frac{1}{2}}\left[\frac{8(1-\mu^2)}{3+\mu}\sin 2\theta\,\beta^2\ln\beta + \beta^2\left\{-\sin 2\theta\left\{\frac{2\pi}{9} + \frac{4}{3}\frac{2\mu^2+3\mu+9}{3+\mu} + \frac{8(\mu-1)^2}{3+\mu}\ln\frac{\gamma}{\sqrt{2}}\right\} + \frac{2}{3}\frac{\mu^2-10\mu+9}{3+\mu}\sin 4\theta\right\} + \pi\beta^4\ln\beta\sin 2\theta\frac{\mu(22+10\mu)}{9(3+\mu)} + \pi\beta^4\left\{-\frac{\sin 2\theta}{148(3+\mu)^2}\left\{248\mu^3+128\mu^2-1405\mu+1329+48(\mu-1) \times(\mu^2+28\mu+59)\ln\frac{\gamma}{\sqrt{2}}\right\} + \frac{71\mu^2-137\mu+62}{36(3+\mu)}\sin 4\theta\right\} + \varepsilon\beta^2\left\{\sin 5\theta\frac{115\mu^2-144\mu-259}{15(3+\mu)} + \sin 3\theta\frac{-\mu^2+60\mu+61}{3(3+\mu)} - (4\mu+5)\sin\theta\right\}\right]. \tag{22}$$

9. DISCUSSION

The membrane and bending solutions have been obtained to an accuracy of β^4 order. Figures 2–5 represent the membrane and bending stress distribution at the hole. The different parameters occurring in the stresses have been taken as follows:

$$\frac{R_0}{h} = 100.$$

For this R_0/h ratio, two different R_0/r_0 ratios have been taken ($R_0/r_0 = 20$ and 30) to study the effect of β. The β values corresponding to the above R_0/r_0 ratios are 0·322 and 0·215. For each value of β, different values of ε are taken corresponding to the semi-cone angle of 0°, 30° and 45°. One important conclusion can be drawn from Figs. 2–5 that the variation of stresses in the conical shell from cylindrical shell decreases as β decreases. For very low value of β, the conical shell and cylindrical shell solutions are not much different from each other. In these computations μ value has been assumed to 0·3.

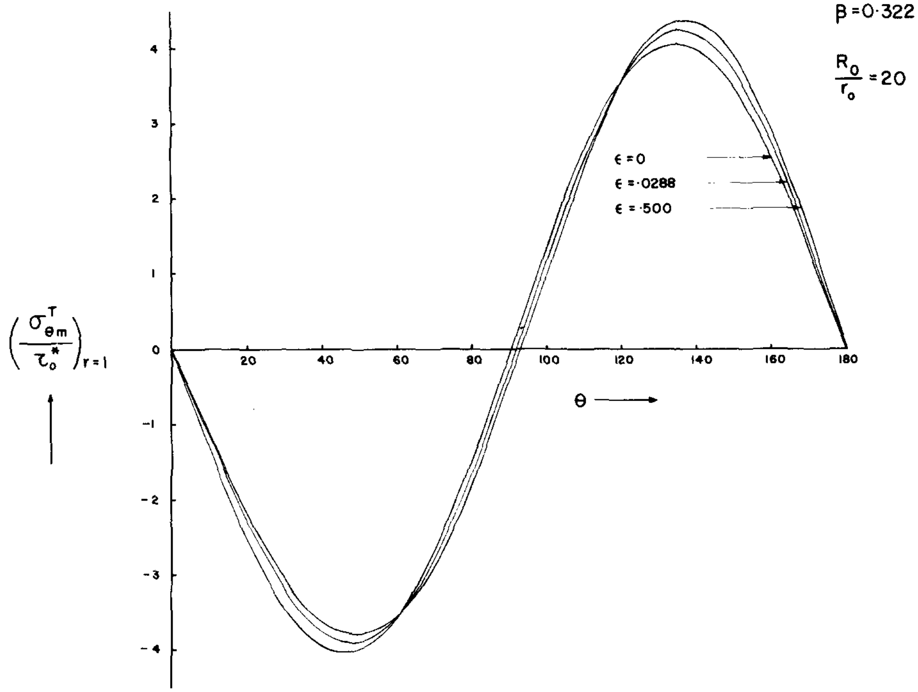

FIG. 2. Membrane stresses due to torsional load.

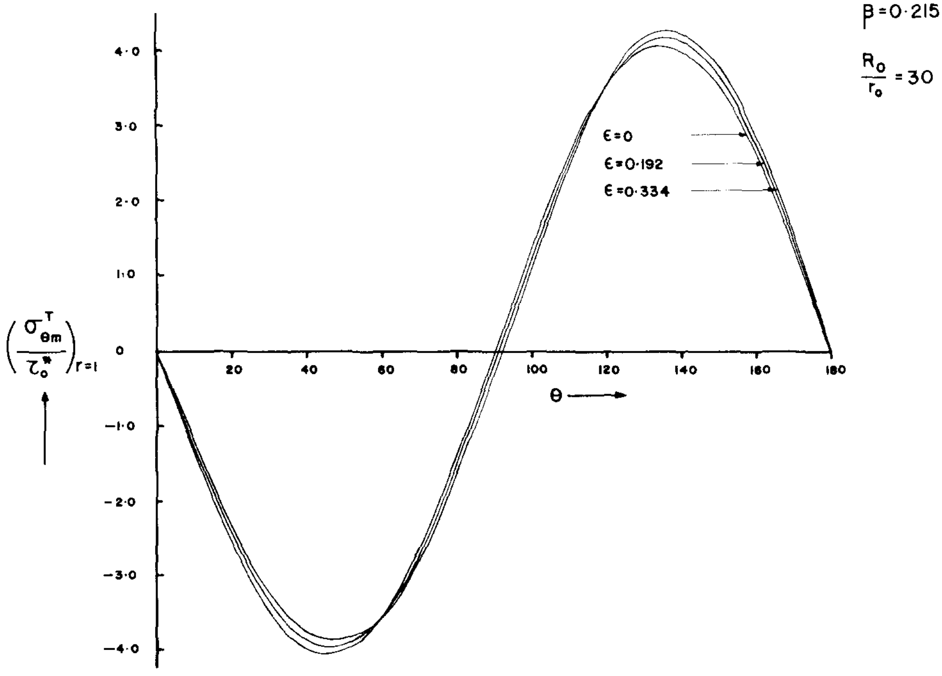

FIG. 3. Membrane stresses due to torsional load.

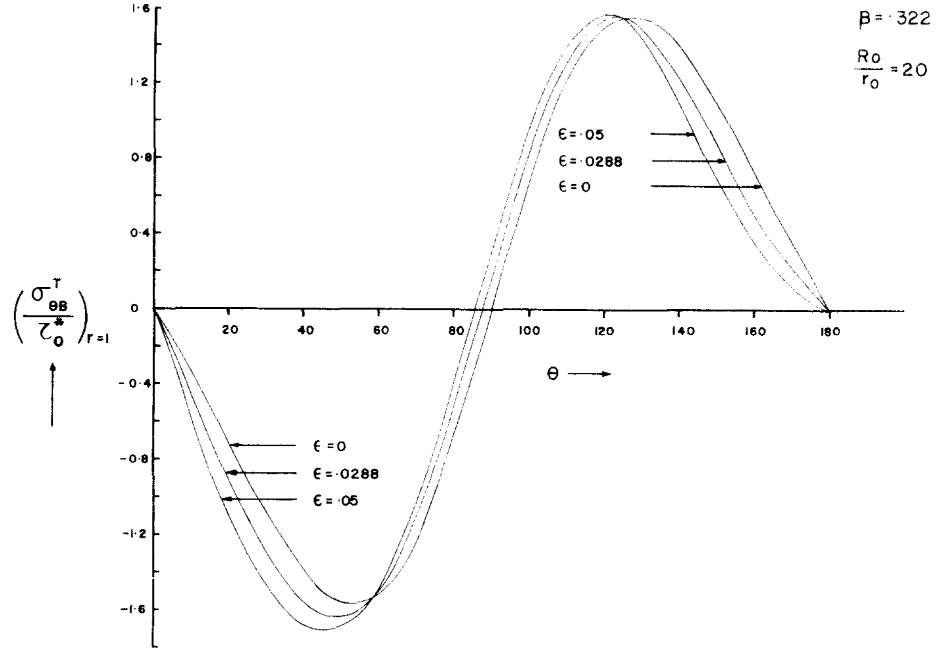

FIG. 4. Bending stresses due to torsional load.

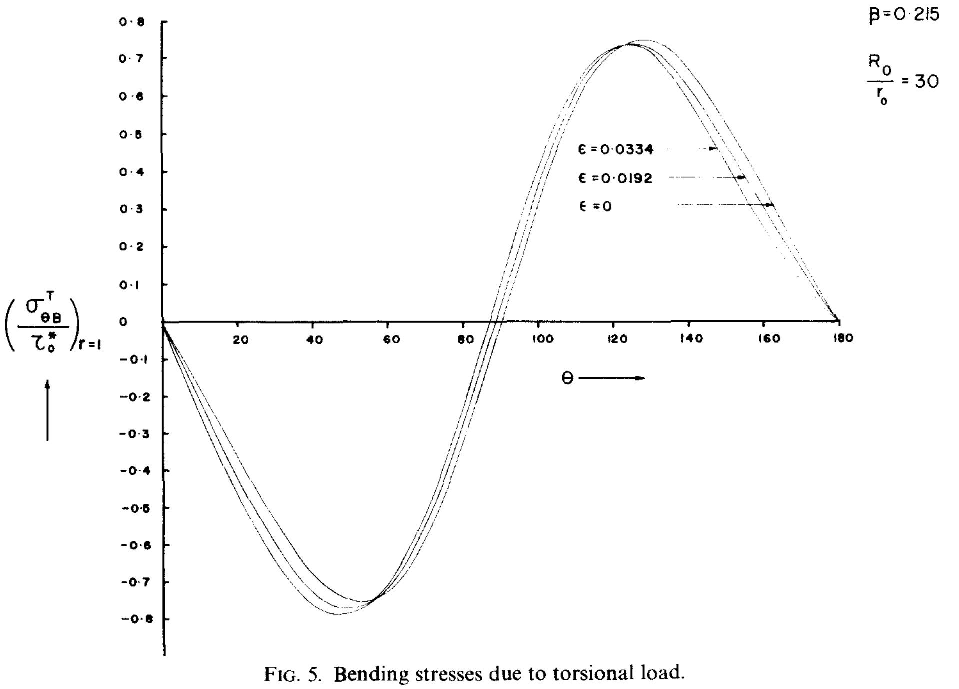

FIG. 5. Bending stresses due to torsional load.

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REFERENCES

[1] B. Basava Raju, M. V. V. Murthy and Ramesh Chandra, Stresses around a circular hole in a shallow conical shell. *Int. J. Solids Struct.* **4**, 993–1011 (1968).
[2] A. I. Lurie, *Statics of Thin-Walled Elastic Shells.* State Publishing House of Technical and Theoretical Literature (1947).
[3] A. N. Guz, (Kiev), Stress in a conical shell weakened by a circular hole. NAL Technical Translation, TT-1.

APPENDIX

Derivation of basic stresses $\bar{N}_r$, $\bar{N}_\theta$ and $\bar{N}_{r\theta}$

Considering the section of a conical shell in the second principal direction subjected to torsion, it can be shown by equation of equilibrium that the membrane stresses are given by

$$\begin{aligned} \bar{N}_{s\psi} &= \tau_0/(1+2\varepsilon r\cos\theta+\varepsilon^2 r^2) \\ \bar{N}_s &= 0, \bar{N}_\psi = 0 \end{aligned} \tag{A-1}$$

where $\tau_0 = T/(2\pi R_0^2)$, T is applied torsional load

Now $\bar{N}_r$, $\bar{N}_\theta$ and $\bar{N}_{r\theta}$ will be computed from the above stresses by using the transformations

$$\begin{aligned} \bar{N}_r &= \bar{N}_{s\psi}\sin 2\lambda \\ \bar{N}_\theta &= -\bar{N}_{s\psi}\sin 2\lambda \\ \bar{N}_{r\theta} &= \bar{N}_{s\psi}\cos 2\lambda \end{aligned} \tag{A-2}$$

where λ is the angle between s and r direction [Fig. 1(a)]. It can be shown that

$$\begin{aligned} \sin 2\lambda &= \frac{\sin 2\theta+2\varepsilon r\sin\theta}{1+2\varepsilon r\cos\theta+\varepsilon^2 r^2} \\ \cos 2\lambda &= \frac{\cos 2\theta+2\varepsilon r\cos\theta+\varepsilon^2 r^2}{1+2\varepsilon r\cos\theta+\varepsilon^2 r^2}. \end{aligned} \tag{A-3}$$

By substituting (A-3) and (A-1) in (A-2), we obtain

$$\begin{aligned} \bar{N}_r &= \tau_0(\sin 2\theta-2\varepsilon r\sin 3\theta+3r^2\varepsilon^2\sin 4\theta) \\ \bar{N}_\theta &= -\tau_0(\sin 2\theta-2\varepsilon r\sin 3\theta+3r^2\varepsilon^2\sin 4\theta) \\ \bar{N}_{r\theta} &= \tau_0(\cos 2\theta-2\varepsilon r\cos 3\theta+3r^2\varepsilon^2\cos 4\theta) \end{aligned}$$

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Абстракт—Даются аналитические решения для напряжений, в конической оболочке с круглым отверстием, на ее горизонтальной поверхности. Оболочка подвержена нагрузке кручения. Метод расчета вызывает возмущени в параметрах, определяющих кривизну и угол наклона оболочки (соответственно β и ϵ). Задерживая члены порядка β^4 и ϵ^2, получаются напряжения в безмоментном состоянии и с учетом изгиба.